

The Morphology of the Thermal Sunyaev-Zel’dovich Sky

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ABSTRACT

At high angular frequencies, beyond the damping tail of the primary CMB power spectrum, the thermal Sunyaev-Zel’dovich (tSZ) effect constitutes the dominant signal in the CMB sky. The tSZ effect is caused by large scale pressure fluctuations in the baryonic distribution in the Universe so its statistical properties can provide estimates of corresponding properties of the projected 3D pressure fluctuations. The power spectrum of the tSZ is a sensitive probe of the amplitude of density fluctuations, and the bispectrum can be used to separate the bias associated with the pressure. The bispectrum is typically probed with its one-point real-space analogue, the skewness. In addition to the ordinary skewness the morphological properties, as probed by the well known Minkowski Functionals (MFs), also require the generalized one-point statistics, which at the lowest order are identical to the generalized skewness parameters. The concept of generalized skewness parameters can be further extended to define a set of three associated generalized *skew-spectra*. We use these skew-spectra to probe the morphology of the tSZ sky or the *y*-sky. We show how these power spectra can be recovered from the data in the presence of an arbitrary mask and noise templates using the well known Pseudo- C_l (PCL) approach for arbitrary beam shape. We also employ an approach based on the halo model to compute the tSZ bispectrum. The bispectrum from each of these models is then used to construct the generalized skew-spectra. We consider the performance of an all-sky survey with Planck-type noise and compare the results against a noise-free ideal experiment using a range of smoothing angles. We find that the skew-spectra can be estimated with very high signal-to-noise ratio from future frequency cleaned tSZ maps that will be available from experiments such as Planck. This will allow their *mode by mode* estimation for a wide range of angular frequencies l and will help us to differentiate them from various other sources of non-Gaussianity.

Key words: : Cosmology–Sunyaev Zeldovich Surveys – Methods: analytical, statistical, numerical

1 INTRODUCTION

Measurements of cosmological parameters from Cosmic Microwave Background (CMB) surveys such as those performed by satellite missions like WMAP¹ and Planck² can not only provide a very accurate picture of the background geometry and dynamics of the Universe but can also reveal much more detailed information about ongoing physical processes. Indeed, with all-sky coverage and wide range of previously uncharted

¹ <http://wmap.gsfc.nasa.gov/>

² <http://www.rssd.esa.int/Planck>

frequencies the data provided by experiments such as Planck will produce “secondary” science which is arguably as valuable as the “primary” science; see e.g. Aghanim, Majumdar, Silk (2008) for a recent review.

The large-scale properties of hot intergalactic gas can be probed through the multifrequency observations. Inverse-Compton scattering of CMB photons known as thermal Sunyaev-Zel’dovich effect (tSZ) leaves a characteristic distortion pattern in the CMB spectrum Sunyaev & Zeldovich (1972, 1980); Birkinshaw (1999); Rephaeli (1995). The fluctuation of this distortion across the sky as probed by CMB observations can provide valuable clues to the fluctuations of the gas density and temperature. In the (low frequency) Rayleigh-Jeans (RJ) regime it produces constant decrement, while there is an increment at high frequencies; in between there is a null (around 217GHz). This characteristic behavior is a potential tool for the separation of tSZ from the other contributions to the temperature anisotropy. Based on accurate knowledge of the tSZ and CMB spectrum, foreground removal techniques have been developed to isolate the tSZ signal in the presence of primary anisotropy and instrumental noise. These techniques are extremely effective in the subtraction of primary anisotropies due to its well understood (perfect black body) frequency dependence and almost exactly Gaussian statistical behavior (Leach 2008; Bouchet & Gispert 1999; Delabroullie, Cardoso & Patanchon 2003). The tSZ effect is now routinely imaged in massive individual galaxy clusters where the temperature of the scattering medium can reach 10keV. This in effect produces a change in CMB temperature of order 1mK at RJ wavelengths. Individual galaxy cluster tSZ images have a variety of astrophysical and cosmological applications including direct measurement of the angular diameter distance to the cluster through a combined analysis of X-ray data and measurement of the gas mass which can be useful in estimation of baryon fraction of the Universe. The High Frequency Instrument (HFI) of Planck, in particular, has been designed with bands centered at the minimum, the null, and the maximum of the thermal SZ (tSZ) emission. The extraction of frequency cleaned CMB and tSZ maps of a catalog of galaxy clusters selected by their tSZ effect are part of the scientific program of Planck. Here we are however also interested in the general intergalactic medium (IGM) where the gas is expected to be at $\leq 1\text{keV}$ in mild overdensities which leads to CMB contributions in the μK range. In this work we primarily focus on the statistical study of wide-field CMB data where tSZ effects lead to anisotropies in the temperature distribution due to both resolved and unresolved galaxy clusters, keeping in mind that the thermal tSZ contribution is the dominant signal beyond the damping tail of the primary anisotropy power spectrum. We primarily focus on analytical modelling morphological properties of the tSZ fluctuations using Minkowski Functionals (MFs).

Modelling of the lower order statistics of the tSZ effect can be performed using either analytical or numerical approaches. Authors using analytical approaches (Seljak 2000; Zhang & Pen 2001; Komatsu & Seljak 2001; Zhang & Seth 2007; Cooray 2000, 2001) have generally used the halo model (Cooray & Seth 2002). To trace the tSZ effect due to photo-ionized gas outside collapsed halos, see e.g. Cooray (2001). A second-order perturbative formalism is used to model the bispectrum in this approach. The gas will typically be at a temperature similar to the ionization energy of hydrogen and helium. The bias associated with the pressure fluctuations is assumed to be redshift dependent. For tSZ effect from material within collapsed halos the shock-heated gas is typically assumed to be in hydrodynamic equilibrium in virialized halos. The statistical description of halos that include the number count distribution are assumed to be described by Press-Schechter formalism (Press & Schechter 1974). The radial profile of such halos are assumed to be that of NFW (Navarro, Frenk & White 1996). These ingredients are sufficient for modeling of tSZ effect from collapsed halos (Cooray 2000).

In addition to analytical modeling the numerical simulation of tSZ plays an important role in our understanding of the physics involved (Persi et al. 1995; Refregier et al. 2000; Seljak et al. 2001; Springel et al. 2001; White, Hernquist & Springel 2002; Lin et al. 2004; Zhang et al. 2004; Cao, Liu & Fang 2007; Roncarelli et al. 2007; Hallman et al. 2009, 2007; da Silva et al. 1999). Some of these studies incorporate complications from additional gas physics such as radiative cooling, preheating and SN/AGN feedback, at least to a certain extent. The inputs are otherwise difficult to incorporate in any analytical calculations. On the other hand the simulations are limited in their dynamic range and can benefit from analytical insights.

The tSZ power spectrum is known to be a sensitive probe of the amplitude of density fluctuations. Higher order statistics, such as the skewness or the bispectrum we study here, can provide independent estimates of the bias associated with the baryonic pressure. Typically a collapsed three-point statistics such as the (one-point) skewness is employed for this purpose. The skewness compresses all available information in the bispectrum. The recently proposed skew-spectrum is a power spectrum associated with the bispectrum (Munshi & Heavens 2010) which is useful to probe non-Gaussianity as a function of scale or the harmonics l . In addition to the lower-order multispectra, morphological statistics such as the Minkowski Functionals (MFs) carry independent information of non-Gaussianity and have been studied extensively in the literature (Hikage et al. 2008, 2006; Hikage, Taruya & Suto 2003; Hikage et al. 2002, 2003). At the lowest order the MFs are completely described by a set of three different skewness parameters that describe complete set of three Minkowski functionals in 2D. We extended these generalized skewness parameters to their corresponding skew-spectra. These skew-spectra sample the bispectra with varying weights and carry independent information. The estimation of these skew-spectra can be done at relatively modest computational cost. The skew-spectra can be constructed by cross-correlating suitable maps that are constructed from the original (frequency cleaned) tSZ maps. The construction involves differential operations on beam smoothed pixelised maps that can also be performed in the harmonic domain. The estimators for skew-spectra can be constructed following the general principle of

power spectrum estimation. We use the well known Pseudo C_l s (PCL) approach developed by (Hivon et al. 2002). It can handle arbitrary sky coverage and arbitrary noise characteristics and beam patterns. We use the PCL approach to construct the variance in the estimators as well as to compute their cross-correlations. These results are accurate for surveys with all-sky coverage and can also be modified to take into account partial sky coverage, using the flat sky approximation.

To relate to the experimental scenarios we will consider an experimental setup similar to the ongoing all-sky experiment Planck (The Planck Collaboration 2006). We consider the range of harmonics (2, 2000). However, the results presented here are also applicable to smaller surveys such as Arcminute Cosmology Bolometer Array Receiver (ACBAR; Runyan et al. (2003))³ which covers the ℓ -range (2000, 3000). ACBAR is a multifrequency millimeter-wave receiver designed for observations of the cosmic microwave background and the Sunyaev-Zel'dovich effect. The ACBAR focal plane consists of a 16 pixel, background-limited, 240 mK bolometer array that can be configured to observe simultaneously at 150, 220, 280, and 350 GHz with 4'-5' FWHM. Together with Planck these two experiments will cover the entire range of ℓ values up to 3000. In addition to Planck we also consider a noise free ideal set up for the range of ℓ values (2, 2000) as a reference.

The paper is organized as follows. In §2 we review the details of the analytical models involving the power spectrum and the bispectrum of the tSZ effect. In §3 we present the skewness parameters that can be used to study the Minkowski Functionals. In §4 we define the triplets of Generalized Minkowski Functionals. We propose the use of skew-spectra associated with individual generalised skewness parameters that carry more information than the ordinary MFs. We show how these skew-spectra are associated with the MFs and are related to the generalised skewness parameters. Next in §5 we present the estimators that can be used to extract the skew-spectra from a noisy data set in the presence of observational mask and arbitrary beam. We also obtain the signal-to-noise for these skew-spectra for realistic observational scenarios. Finally, §6 is devoted to a discussion of our results and future prospects.

The particular cosmology that we will adopt for numerical study is specified by the following parameter values (to be introduced later): $\Omega_\Lambda = 0.741$, $h = 0.72$, $\Omega_b = 0.044$, $\Omega_{\text{CDM}} = 0.215$, $\Omega_M = \Omega_b + \Omega_{\text{CDM}}$, $n_s = 0.964$, $w_0 = -1$, $w_a = 0$, $\sigma_8 = 0.803$, $\Omega_\nu = 0$.

2 LOWER ORDER STATISTICS OF THE TSZ EFFECT

In this section we briefly review the two different approaches that are commonly used to model the tSZ effect (for more details see (Cooray, Baumann & Sigurdson 2005) and the references there in). We will use these models later to study the morphological properties of the SZ effect. The tSZ temperature fluctuation $\Theta^{\text{SZ}}(\hat{\Omega}, \nu) = \delta T(\hat{\Omega}, \nu)/T_{\text{CMB}}$ is given by the (opacity-weighted) pressure fluctuations i.e.

$$\Theta^{\text{SZ}}(\hat{\Omega}, \nu) = \delta T(\hat{\Omega})/T_{\text{CMB}} \equiv g_\nu(x)y(\hat{\Omega}) = g_\nu(x) \int_0^{r_0} d\eta a(\eta) \sigma_T/m_e \pi_e(\hat{\Omega}, \eta); \quad (1)$$

we will use the symbol \mathbf{x} to represent a comoving coordinate. The electron pressure is denoted as $\pi_e = n_e k_B T_e$. Here $y(\hat{\Omega})$ is the Compton y -parameter, σ_T is the Thomson cross-section, k_B is the Boltzman constant, m_e is the electron rest mass, n_e is the electron number density and T_e the electron temperature. The function $g_\nu(x)$ encodes the frequency dependence of the tSZ anisotropies. It relates the temperature fluctuations at a frequency ν with the Compton parameter y , i.e. $g_\nu(x) = x \coth(x/2) - 4$ and $x = (h\nu/k_B T_{\text{CMB}}) = \nu/(56.84 \text{ GHz})$. In the low frequency part of the spectrum $g_\nu(x) = -2$, for $x \ll 1$, here x is the dimensionless frequency as defined above. Unless stated otherwise all our results will be for the low frequency limiting case. The conformal time η can be expressed in terms of the cosmological density parameters Ω_M , Ω_K and Ω_Λ by the following expression: $\eta(z) = \int_0^z dz'/H(z')$ and $E^2(z) = H^2(z)/H_0^2 = [\Omega_M(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda]$. Introducing the τ the Thompson optical depth and the fractional fluctuations in pressure as $\pi = \delta p_e/\rho_e$ we can write $\Theta^{\text{SZ}}(\hat{\Omega}, \nu) = \int_0^{r_0} dr \dot{\tau} \pi(\hat{\Omega}, r)$. Here r is the comoving distance, τ is the Thomson optical depth, and overdots represent derivatives with respect to r .

We will primarily be working in the Fourier domain. The three-dimensional electron pressure π can be decomposed into its Fourier coefficients using the following convention that we will follow: $\delta\pi(\mathbf{x}) = (\pi(\mathbf{x}) - \langle\pi(\mathbf{x})\rangle)/\langle\pi(\mathbf{x})\rangle$ with $\delta\pi(\mathbf{k}) = \int d^3\mathbf{x} \delta\pi(\mathbf{x}) \exp[-i\mathbf{k} \cdot \mathbf{x}]$. The projected statistics that we will consider can be related to the 3D statistics which are defined by the following expressions which specifies the power spectrum $P_\pi(k)$, the bispectrum $B_\pi(k_1, k_2, k_3)$ and the trispectrum $T_\pi(k_1, k_2, k_3, k_4)$ in terms of the Fourier coefficients:

$$\langle\delta\pi(\mathbf{k}_1, r_1)\delta\pi(\mathbf{k}_2, r_2)\rangle_c = (2\pi)^3 P_\pi(k_1; r_1, r_2) \delta_{3D}(\mathbf{k}_1 + \mathbf{k}_2) \quad (2)$$

$$\langle\delta\pi(\mathbf{k}_1, r_1)\delta\pi(\mathbf{k}_2, r_2)\delta\pi(\mathbf{k}_3, r_3)\rangle_c = (2\pi)^3 B_\pi(k_1, k_2, k_3; r_i) \delta_{3D}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \quad (3)$$

$$\langle\delta\pi(\mathbf{k}_1, r_1)\delta\pi(\mathbf{k}_2, r_2)\delta\pi(\mathbf{k}_3, r_3)\delta\pi(\mathbf{k}_4, r_4)\rangle_c = (2\pi)^3 T_\pi(k_1, k_2, k_3, k_4; r_i) \delta_{3D}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \quad (4)$$

³ <http://cosmology.berkeley.edu/group/swlh/acbar/>

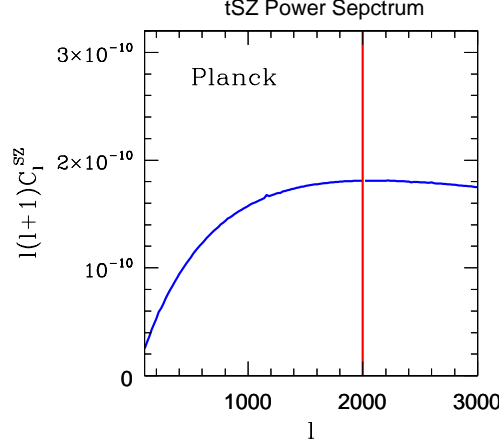


Figure 1. The power spectrum of SZ is depicted for our choice of cosmological parameters. The halo model was used for the computation of the power spectrum. See section §2.3 for a short discussion and relevant parameter values. For an all-sky Planck type experiment we consider the range $l = (2, 2000)$

Our aim is to relate the statistics of pressure fluctuations $\delta\pi(\mathbf{x})$ to those of the underlying density fluctuations $\delta(\mathbf{x})$. The hierarchy of higher-order correlation functions can be defined using a similar set of equations. The corresponding multispectra will be denoted with a subscript δ i.e. B_δ and T_δ will represent the bi- and trispectrum of the underlying density contrast and the power spectrum will be denoted as P_δ . The δ_{3D} functions represent the 3D Dirac delta function and represent translational invariance of corresponding correlation hierarchy in real space. The unequal time correlators appearing in the expressions of the multispectra will typically collapse to equal time correlators due to the use of Limber approximations that we will discuss next.

2.1 Linking projected statistics with 3D statistics

The spherical harmonics decomposition of the projected field $\Theta^{SZ}(\hat{\Omega})$ will be represented as Θ_{lm}^{SZ} can be expressed in terms of a weight $W^{SZ}(r)$. We will henceforth suppress the frequency ν dependence of a_{lm}^{SZ} as well as $\Theta^{SZ}(\hat{\Omega})$. The line-of-sight integration over r projects the 3D $\delta_{lm}(\mathbf{k}, r)$ onto the harmonics a_{lm}^{SZ} :

$$\Theta_{lm}^{SZ} = \int d\hat{\Omega} Y_{lm}^*(\hat{\Omega}) \Theta^{SZ}(\hat{\Omega}) = \sqrt{\frac{2}{\pi}} \int_0^{r_0} W^{SZ}(r) \int k^2 dk \sum_{lm} j_l(kr) Y_{lm}^*(\hat{\Omega}) \delta_{lm}(\mathbf{k}, r); \quad W^{SZ}(r) = -2b_\pi(r)\dot{\tau}; \quad \dot{\tau} = c\sigma_T n_e(z) \quad (5)$$

Here $Y_{lm}(\hat{\Omega})$ represents a spherical harmonic and $j_l(kr)$ is the spherical Bessel function of order l . The weight $W^{SZ}(r)$ is introduced by projecting the 3D statistics onto the sky. We use the following relation to relate $\delta_{lm}(\mathbf{k}, r)$ in the spherical basis with $\delta(\mathbf{k}, r)$ in Fourier basis to obtain that

$$\delta_{lm}(\mathbf{k}) = (2\pi)^{-3/2} \int d\hat{\Omega} Y_{lm}(\hat{\Omega}) \delta(\mathbf{k}) \quad (6)$$

. We can use this expression next in Eq.(5) and use the definition of power spectrum from Eq.(2). Using the integral representation of the Dirac's delta function $\delta_{3D}(\mathbf{k}) = (2\pi)^{-3} \int \exp(i\mathbf{k} \cdot \mathbf{x}) d^3\mathbf{x}$; in association with the Rayleigh's expansion of the plane wave in terms of spherical harmonics $\exp(i\mathbf{k} \cdot \mathbf{x}) = 4\pi \sum_{lm} i^l j_l(kx) Y_{lm}(\hat{\Omega}_k) Y_{lm}(\hat{\Omega})$. Using these expression we arrive at the following expression which projects the 3D power spectrum $P(k, r_1, r_2)$ to the spherical sky (Cooray 2001):

$$C_l^{SZ} = \frac{2}{\pi} \int W^{SZ}(r_1) dr_1 \int W^{SZ}(r_2) dr_2 \int dk k^2 j_{l_1}(kr_1) j_{l_2}(kr_2) P_\pi(k, r_1, r_2) \quad (7)$$

The following form of Limber's approximation (Limber 1954; LoVerde & Afshordi 2008) is remarkably accurate at large angular scales: $l \leq 100$: $\int F(k) k^2 dk j_{l_1}(k_1 r) j_{l_2}(k_2 r) = (\pi/2r^2) F(l/r) \delta_D(r_1 - r_2)$. The use of this form of Limber's approximation greatly simplifies the subsequent

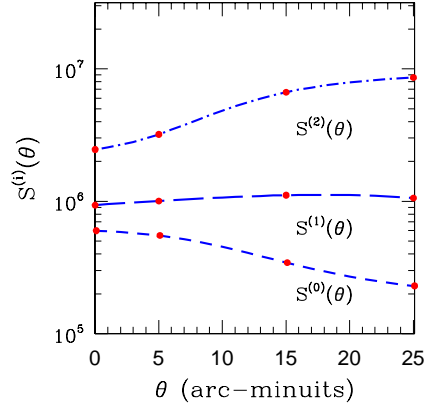


Figure 2. The three one-point skewness parameters $S^{(0)}$ (short-dashed), $S^{(1)}$ (long-dashed) and $S^{(2)}$ (dot-dashed) are plotted as functions of the FWHM θ for a Gaussian beam. The results plotted are for ideal surveys without instrumental noise. The skewness parameter $S^{(0)}$ is the ordinary skewness describing the departure from non-Gaussianity at the lowest order. The points are results from numerical computations using the halo model and lines denote smooth fits to the results. The one point skewness parameters $S^{(i)}$ is related to the skew-spectra $S_l^{(i)}$ through the following expression: $S^{(i)} = \sum_l (2l+1) S_l^{(i)}$.

results by reducing the correlator of multiple time slices to single time correlators (Cooray 2001).

$$\mathcal{C}_l^{\text{SZ}} = \langle \Theta_{lm}^{\text{SZ}} \Theta_{lm}^{\text{SZ}*} \rangle = \int_0^{r_0} dr \frac{[W^{\text{SZ}}(r)]^2}{d_A^2(r)} P_\pi \left(\frac{l}{d_A(r)}; r \right) \quad (8)$$

Here $d_A(r)$ is the angular diameter distance in terms of the comoving radial distance r . The Power spectrum $\mathcal{C}_l^{\text{SZ}}$ is plotted in Fig-1 as a function of the harmonics l .

The bispectrum in the harmonic domain is represented by the following three-point correlation function:

$$B_{l_1 l_2 l_3}^{\text{SZ}} \equiv \sum_{mm'm''} \begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix} \langle \Theta_{lm}^{\text{SZ}} \Theta_{l'm'}^{\text{SZ}} \Theta_{l''m''}^{\text{SZ}} \rangle_c \quad (9)$$

The matrix above in Eq.(9) represents the Wigner $3j$ symbols (Edmonds 1968) and $B_{l_1 l_2 l_3}^{\text{SZ}}$ defines the tSZ bispectrum in the spherical sky. The equation linking the projected bispectrum $B_{l_1 l_2 l_3}^{\text{SZ}}$ and its 3D analogue $B_\pi(k_1, k_2, k_3)$, defined in Eq.(3), can be derived following exactly the same procedure. We quote here the final expression which takes the following form (Cooray 2001):

$$B_{l_1 l_2 l_3}^{\text{SZ}} = I_{l_1 l_2 l_3} \int dr \frac{[W^{\text{SZ}}(r)]^3}{d_A^3(r)} B_\pi \left(\frac{l_1}{d_A(r)}, \frac{l_2}{d_A(r)}, \frac{l_3}{d_A(r)}; r \right); \quad I_{l_1 l_2 l_3} = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \quad (10)$$

Next we describe models for the 3D power spectrum $P_\pi(k; r)$ and bispectrum $B_\pi(k, k', k''; r)$ that will be used for computation of the projected SZ versions $\mathcal{C}_l^{\text{SZ}}$ and $B_{l_1 l_2 l_3}^{\text{SZ}}$. Modelling of the tSZ statistics requires a model for the clustering of underlying dark matter distributions δ . We will employ extensions of perturbative calculations as well as halo model predictions to model the underlying statistics of dark matter clustering. We will describe these model in following subsections. We have shown the angular power spectrum $\mathcal{C}_l^{\text{SZ}}$ for SZ effect in Figure-1 as a function of harmonic number l . For the computation of this power spectra a halo model approach was used which will be discussed in §2.3. As we discussed in the Introduction, modelling the tSZ statistics involves modeling the underlying dark matter clustering and its relationship with the baryonic clustering and thence the fluctuation in baryonic pressure. This modeling has so far been performed using two complimentary approach. In the simpler approach the clustering of dark matter is described using second order perturbation theory or its extensions. The clustering of baryons are described using a biasing scheme. These inputs are sufficiently accurate especially at larger length scales to model the SZ sky. We will consider the linear biasing model first.

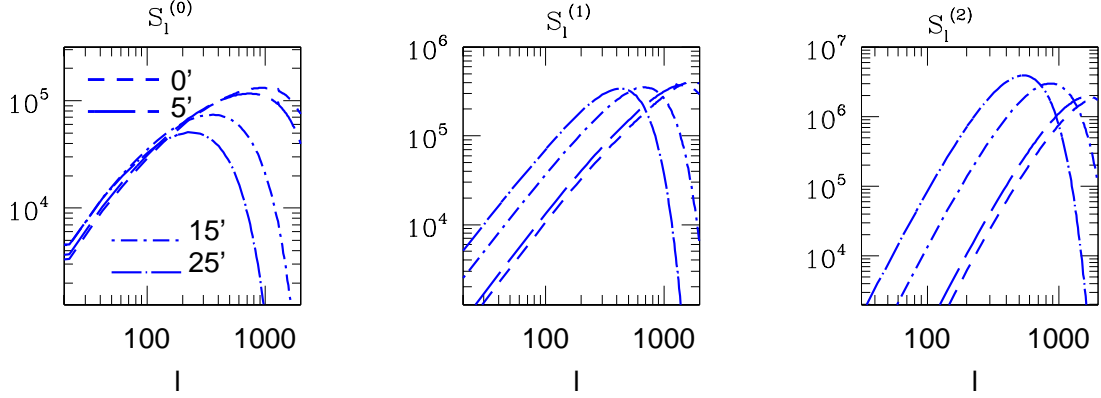


Figure 3. The absolute values of skew-spectra are plotted as a function of the harmonic l . From left to right the plots correspond to $S_l^{(0)}$, $S_l^{(1)}$ and $S_l^{(2)}$ respectively. We consider four different beams with FWHM = $0'$, $5'$, $15'$ and $25'$ respectively as indicated. These results correspond to *ideal noise-free* experiments and the resolution is taken to be $l_{max} = 2000$. The results are obtained using halo model prescription.

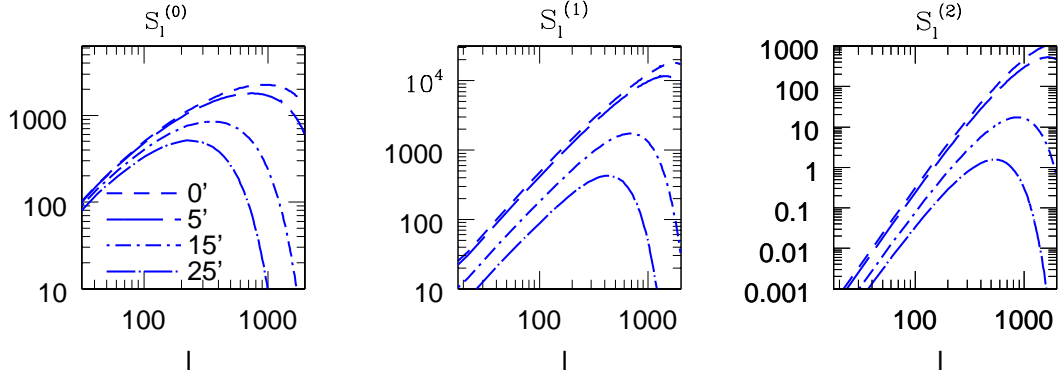


Figure 4. The absolute values of skew-spectra are plotted as a function of the harmonic l . From left to right the plots correspond to $S_l^{(0)}$, $S_l^{(1)}$ and $S_l^{(2)}$ respectively. We consider four different beams with FWHM = $0'$, $5'$, $15'$ and $25'$ respectively as indicated. These results include Planck-type noise and correspond to an all-sky survey.

2.2 Linear Biasing and Perturbative Approach

In a perturbative approach the gravitational dynamics is analyzed by expanding the large scale density field δ in a perturbative series and a redshift dependent linear biasing is assumed (Goldberg & Spergel 1999a,b). It is valid as long as the variance of the density contrast is smaller than unity and such a treatment is suitable for diffuse tSZ component (Hansen et al. 2005). For large smoothing scales such calculations can provide valuable insight to gravitational dynamics. The perturbative bispectrum $B_\delta(k_1, k_2, k_3; r)$ for the density contrast δ from such calculations is given by (Peebles 1971):

$$B_\delta(k_1, k_2, k_3; r) = F_2(k_1, k_2)P_\delta(k_1, r)P_\delta(k_2, r) + \text{cyc.perm.} \quad (11)$$

$$F_2(k_1, k_2) = 1 - \frac{2}{7}\Omega_M^{-2/63} + \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{2k_1k_2} \left(\frac{k_1}{k_2} + \frac{k_2}{k_1} \right) + \frac{2}{7}\Omega_M^{-2/63} \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{k_1^2k_2^2} \quad (12)$$

In the deeply nonlinear regime we do not have a complete analytical model of gravitational clustering. Typically, plausible approximations that include the halo model (to be discussed next) are used for this purpose, with varying degree of success. One such ansatz is the well known *hierarchical ansatz* that assumes that the higher order correlation functions can be constructed from the products of two point correlation functions (Bernardeau et al 2002). The different tree diagrams thus constructed are all represented by various *hierarchical amplitudes* which are left arbitrary. The bispectrum in such a hierarchical scenario takes the following form:

$$B_\delta(k_1, k_2, k_3; r) = Q_3 [P(k_1)P(k_2) + \text{cyc.perm.}]; \quad Q_3(n) = [4 - 2^n]/[1 + 2^{n+1}]. \quad (13)$$

The expression for $Q_3(n)$ is adopted from (Scoccimarro & Frieman 1999). For a generic power spectrum one can replace n with the local linear power spectral index at $(k_1 + k_2 + k_3)/3$ (Hui 1999). More elaborate schemes that interpolate the quasilinear regime and the highly non-linear regime have been devised in recent years Scoccimarro & Couchman (2001). Though we will restrict ourselves to the computation of the bispectrum in this paper, the approach can also be extended beyond third order. The hierarchical ansatz captures the salient features of non-linear clustering; the expressions are relatively simpler than the full perturbative expressions and have been tested in variety of cosmological contexts. In connecting the statistics of the electron pressure field π namely the power spectrum $P_\pi(k; r)$ and $B_\pi(k, k', k''; r)$ we will use the following local biasing model $P_\pi(k, r) = b_\pi(r)^2 P_\delta(k, r)$ and $B_\pi(k, k', k''; r) = b_\pi(r)^3 B_\delta(k, k', k''; r)$. Following (Goldberg & Spergel 1999a,b; Cooray 2000, 2001) the bias is chosen to be of the following form $b_\pi(r) = b_\pi(0)/(1+z)$ with $b_\pi(0) = k_B T_e(0)/(m_e c^2 b_\delta)$. These 3D Fourier statistics will be the ingredients for computation of projected statistics on the surface of the sky using Eq.(8) and Eq.(10).

2.3 Halo Model and tSZ: A brief Review

The halo model description of large scale structure relies on modeling of the clustering of halos and predictions from perturbative calculation to model the non-linear correlation functions. In the context of halo model the dark matter is assumed to be in collapsed halos which are characterized by their mass, A brief Review radial profile, halo occupation numbers and underlying correlation hierarchy. The baryonic fluid is assumed to be in equilibrium with the dark matter profile and is characterized by a specific equation of state. The use of halo model is well established in cosmology and has a long history (Cooray & Seth 2002). It has been used relatively recently is modeling of clustering of galaxies, weak lensing observables and statistics of CMB secondaries. In the context of tSZ it has been used to compute the leading order non-Gaussianity such as bispectrum as well as higher order statistics such as trispectrum which is important in computing the covariance of ordinary power spectrum. It is known that the halo overdensity at a given position \mathbf{x} , $\delta^h(\mathbf{x}, M; z)$ can be related to the underlying density contrast $\delta(\mathbf{x}, z)$ by a Taylor expansion as was shown by (Mo & White. 1996):

$$\delta^h(\mathbf{x}, M; z) = [b_1(M; z)\delta(\mathbf{x}, z) + \frac{1}{2}b_2(M, z)\delta^2(\mathbf{x}, z) + \dots]. \quad (14)$$

The expansion coefficients are functions of the threshold $\nu(M; z) = \delta_c(z)/\sigma(M, z)$. Where δ_c is the threshold for a spherical over-density to collapse and $\sigma(M, z)$ is r.m.s fluctuation within a top-hat filter. The parameter $\delta_c(z)$ is the value of the spherical over-density at which it collapses at a redshift z and is given by the following functional fit for a given Ω_Λ and Ω_M :

$$\delta_c(z) = \frac{3(12\pi)^{2/3}}{20} \left[1 - \frac{5}{936} \ln \left(1 + \frac{1 - \Omega_M}{\Omega_M(1+z)^3} \right) \right] \quad (15)$$

The rms fluctuation in the sphere of radius R is constructed from linearly extrapolated power spectrum $P(k)$ $\sigma^2(M; z) = D^2(z) \int d \ln k \Delta^2(k) |W(kR)|^2$ where $W(x)$ is the to-phat window and $\Delta^2(k)$ is the dimensionless power spectrum is given by $\Delta^2(k) = k^3 P(k)/2\pi^2$. The linear order bias $b_1(M; z)$ introduced in in this expansion of δ_h in terms of underlying density contrast δ depends on the threshold $\nu(M; z)$ (Mo, Jing & White 1997):

$$b_1(M; z) = 1 + \frac{(a\nu^2(M; z) - 1)}{\delta_c(z)} + \frac{2p}{\delta_c(z) \{1 + [a\nu^2(M; z)]^p\}}. \quad (16)$$

The linear growth factor $D(z)$ is defined by the following expression: $D(z) \propto H(z) \int_z^\infty dz' (1+z') [H(z')]^{-3}$ and is normalized to unity at $z = 0$ i.e $D(0) = 1$. The main two ingredients in the halo model are the radial profile of the halos and the number density of halos. The number density is given by: $f(\nu) = \sqrt{2A^2 a^2 / \pi} [1 + (a\nu^2)^{-p}] \exp(-a\nu^2/2)$ and the threshold ν . The parameters (p, a) takes the value $(0, 1)$ for Press-Sechter mass function (Press & Schechter 1974). The constant A is determined by imposing the normalization $\int f(\nu) d\nu = 1$. The halo model incorporates perturbative aspects of gravitational dynamics through the halo-halo correlation hierarchy. The nonlinear features take direct contribution from

the halo profile; the number-density of haloes also needs to be determined. The total power spectrum $P^t(k)$ at non-linear scale can be written as (Seljak 2000) sum of two separate contributions to the power spectrum $P_\pi^t(k, z) = P_\pi^{1h}(k, z) + P_\pi^{2h}(k, z)$. These represent contributions from clustering of dark matter particles in two different halos, i.e. the halo-halo term and a contribution from a single halo or one-halo term P^{1h} . These terms are in turn expressed in terms of two integrals:

$$P_\pi^{1h}(k, z) = I_{2,\pi}^0(k, z); \quad P_\pi^{2h}(k, z) = [I_{1,\pi}^{(0)}(k, z)]^2 P_\delta(k, z) \quad (17)$$

The integrals themselves depend on the number counts of halos as a function of the concentration parameter c and the mass M .

$$I_{2,\pi}^{(0)}(k; z) = \int dM \int dc \frac{d^2 n}{dM dc} |\pi_e(k, |M, c; z)|^2; \quad I_{1,\pi}^{(1)}(k; z) = \int dM \int dc \frac{d^2 n}{dM dc} b_1(M, z) |\pi_e(k, |M, c; z)| \quad (18)$$

The bispectrum can be expressed in an analogous manner:

$$B_\pi^t(k_1, k_2, k_3) = B_\pi^{3h}(k_1, k_2, k_3) + B_\pi^{2h}(k_1, k_2, k_3) + B_\pi^{1h}(k_1, k_2, k_3) \quad (19)$$

$$B_\pi^{1h} = I_3^0(k_1, k_2, k_3); \quad B_\pi^{2h}(k_1, k_2, k_3) = I_2^1(k_1, k_2) I_1^0(k_3) P_\pi(k_3) + cyc.perm. \quad (20)$$

$$B_\pi^{3h} = [2J(k_1, k_2, k_3) I_1^1(k_3) + I_1^2(k_3)] I_1(k_1) I_1(k_2) P_\delta(k_1) P_\pi(k_2) + cyc.perm. \quad (21)$$

The kernel $J(k_1, k_2, k_3)$ is same as $F_2(k_1, k_2, k_3)$ introduced in Eq.(12). We have introduced the following notation above:

$$I_{q,\pi}^{(p)}(k; z) = \int dM \int dc \frac{d^2 n}{dM dc} b_p(M, z) |\pi_e(k, |M, c; z)|^q. \quad (22)$$

The Fourier transform of the halo electron pressure profile $\pi(r)$ is denoted by $\pi_e(k)$ above:

$$\pi_e(k) = \int d^3 \mathbf{r} \pi_e(r) \exp(-i\mathbf{k} \cdot \mathbf{r}) = \int_0^\infty 4\pi r^2 dr \pi_e(r) \frac{\sin(kr)}{kr}. \quad (23)$$

For the dark matter profile we assume $\rho_d(r) \equiv \rho_s u^{-1} (1 + u)^{-2}$ with $u \equiv r/r_s$. Here the dark matter is assumed to be a NFW profile and r_s is a characteristic scale radius. In the context of the spherical collapse model, the outer extent of a cluster is taken to be its virial radius $r_v = [3M/4\pi\rho_M(z)\Delta(z)]$ where $\rho_M(z)$ is the mean background matter density of the Universe at redshift z and $\Delta_M(z)$ is the over-density of the halo relative to the background density. The ratio of the virial radius to the scale radius is called the concentration parameter $c \equiv r_v/r_s$. Together c and M determines the dark matter distribution of a given halo. The gas density profile in terms of the dimensionless parameter u is $\rho_g(u) \equiv \rho_0 \rho_g(u)$. In this particular case a polytropic equation of state is typically used. According to the numerical simulation clusters of a given mass have a range of concentration parameters. The number density of clusters with a given mass M and concentration parameter c are expressed in terms of a bivariate distribution function $d^2 n(M, c; z)/dM dc = dn/dM(M; z) P(c|M; z)$. The conditional probability distribution function of a halo having concentration c for a given mass M in numerical simulations was found to be well approximated by a log-normal distribution. $P(c|M; z)dc = \exp[-(\log c - \log \bar{c})/2\sigma_{\log c}] dc / (c \ln(10))$ with $\bar{c}(M, z) = c_0/1 + z[M/M_*(z=0)]^{-\alpha_c}$. Here M_* is the mass scale at which $\nu(M_*, z) = 1$ and $c_0 = 9$ and $\alpha_c = 0.13$ and $\sigma_{\log c} = 0.14$. More complicated models are possible where a mass dependent $\sigma_{\log c}$ can be incorporated. For more discussion on equilibrium baryonic gas profile and its relation with the underlying halo profile see Cooray, Baumann & Sigurdson (2005).

3 MINKOWSKI FUNCTIONALS

In this section we will review the basics of the Minkowski Functional (MFs) in brief. The MFs are morphological descriptors well known in cosmology. Put simply, morphological properties are the properties that remain invariant under rotation and translation; see e.g. Hadwiger (1959) for a more mathematical discussion. We will concentrate on a set of three MFs defined in 2D that are defined over an excursion set Σ with a boundary $\partial\Sigma$ for a given threshold ν . Following the notation of Hikage et al. (2008) we define:

$$V_0(\nu) = \int_\Sigma da; \quad V_1(\nu) = \frac{1}{4} \int_{\partial\Sigma} dl; \quad V_2(\nu) = \frac{1}{2\pi} \int_{\partial\Sigma} \mathcal{K} dl. \quad (24)$$

Here \mathcal{K} denotes the curvature along the boundary $\partial\Sigma$. The three MFs $V_0(\nu)$, $V_1(\nu)$ and $V_2(\nu)$, correspond to the area of the excursion set Σ , the length of its boundary $\partial\Sigma$ as well as the integral of curvature \mathcal{K} along its boundary which is also related to the genus and hence the

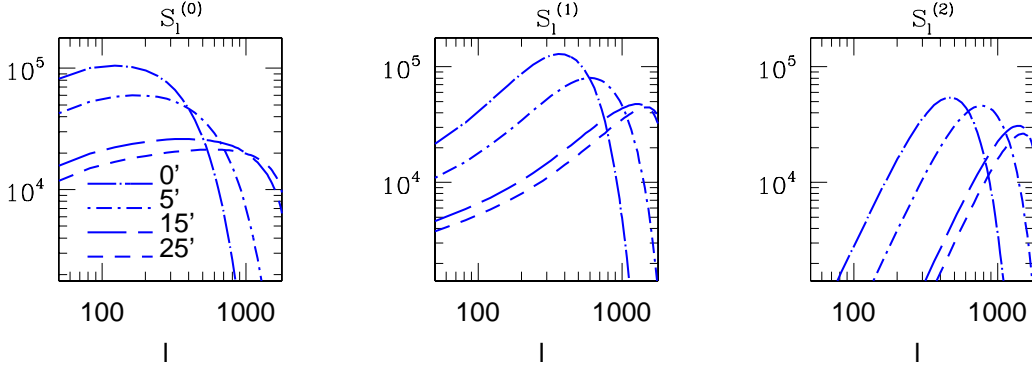


Figure 5. The skew-spectra are plotted as a function of the harmonic l . From left to right the plots correspond to $S_l^{(0)}$, $S_l^{(1)}$ and $S_l^{(2)}$ respectively. We consider four different beams with FWHM = $0'$, $5'$, $15'$ and $25'$ respectively as indicated. These results correspond to an ideal all-sky noise free experiment. A perturbative calculation which assumes redshift dependent linear biasing was used.

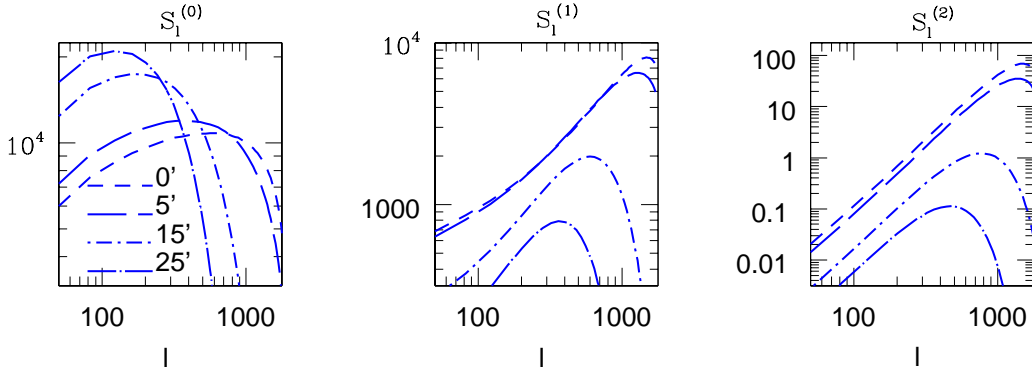


Figure 6. The skew-spectra are plotted as a function of the harmonic l . From left to right the plots correspond to $S^{(0)}$, $S^{(1)}$ and $S^{(2)}$ respectively. We consider four different beams with FWHM = $0'$, $5'$, $15'$ and $25'$ respectively as indicated. These results include Planck-type noise.

Euler characteristics. The MFs can be decomposed into two different contributions Gaussian $V_k^G(\nu)$ and non-Gaussian $\delta V_k(\nu)$ i.e. $V_k(\nu) = V_k^G(\nu) + \delta V_k(\nu)$ with $\nu = \Theta/\sigma_0$ and $\sigma_0^2 = \langle \Theta^2 \rangle$. We non-Gaussian contribution i.e. $\delta V_k(\nu)$. We will further separate out an amplitude A in the expressions of both of these contributions which depend only on the power spectrum of the perturbation through σ_0 and σ_1 (see e.g. Hikage et al. (2008)):

$$V_k^G(\nu) = A \exp\left(-\frac{\nu^2}{2}\right) H_{k-1}; \quad \delta V_k(\nu) = A \exp\left(-\frac{\nu^2}{2}\right) \left[\delta V_k^{(2)}(\nu) \sigma_0 + \delta V_k^{(3)}(\nu) \sigma_0^2 + \delta V_k^{(4)}(\nu) \sigma_0^3 + \dots \right] \quad (25)$$

$$\delta V_k^{(2)}(\nu) = \left[\frac{1}{6} S^{(0)} H_{k+2}(\nu) + \frac{k}{3} S^{(1)} H_k(\nu) + \frac{k(k-1)}{6} S^{(2)} H_{k-2}(\nu) \right]; \quad A = \frac{1}{(2\pi)^{(k+1)/2}} \frac{\omega_2}{\omega_{2-k} \omega_k} \left(\frac{\sigma_1}{\sqrt{2}\sigma_0} \right)^k. \quad (26)$$

The constant ω_k introduced above is the volume of the unit sphere in k dimensions. $w_k = \pi^{k/2}/\Gamma(k/2 + 1)$ in 2D we will only need $\omega_0 = 1$, $\omega_1 = 2$ and $\omega_2 = \pi$. The Hermite polynomials $H_k(\nu)$ appear in the expression of MFs for weakly non-Gaussian random field. The lowest-order departure from Gaussianity $\delta V_k^{(2)}(\nu)$ is encoded in three different generalized skewness parameters $S^{(0)}$, $S^{(1)}$, $S^{(2)}$. The next to leading order

correction terms $\delta V_k^{(2)}(\nu)$ are typically neglected as they are order of magnitude smaller. Just as the generalized three skewness parameters they can be constructed from generalization of kurtosis. All three skewness parameters can be constructed from the bispectrum using different weights to sample different modes. In our next section we will discuss these generalized skewness parameters and introduce the power spectra associated with them. We will denote the power spectra with $S_l^{(0)}$, $S_l^{(1)}$ and $S_l^{(2)}$. These will carry more information than their one-point counterparts the generalized skewness parameters.

Estimation of MFs typically is done directly by employing a grid-based approach in real space, so that the integrals introduced in Eq.(24) are estimated on a discrete set of points. Here we will show that the computation of $S_l^{(i)}$ introduced in Eq.(26) can also be done in harmonic space. This will be particularly suitable for near all-sky surveys where galactic masks and point source masks can be incorporated in the analysis relatively easily. This will allow a cross-comparison of both methods for any systematics as well as residuals from component separation. We will deal with near all-sky surveys but flat-sky results can also be recovered using standard procedure which will be suitable for smaller patch-sky surveys.

The Fig.2 shows the parameters $S^{(0)}$, $S^{(1)}$ and $S^{(2)}$ as a function of FWHM assuming a Gaussian beam. We have assumed a halo model for computation of these generalized skewness parameters. The parameters defined in Eq. (26), σ_0 and σ_1 , appear in the perturbative expansion of the MFs as well as in their normalization. These parameters depend on the level of noise.

4 GENERALISED SKEW-SPECTRA AND MFs

The skew-spectra are cubic statistics that are constructed by cross-correlating two different fields. One of the field used is a composite field typically a product of two maps either in its original form or constructed by means of relevant differential operations. The second field will typically be a single field but may be constructed by using differential operators on the original field. All of these three skewness contribute to the three MFs that we will consider in 2D. The first of the skew-spectra was studied by (Cooray 2001) and was later generalized by Munshi & Heavens (2010) and is related to commonly used skewness. The skewness in this case is constructed by cross-correlating the squared map $[\Theta^2(\hat{\Omega})]$ with the original map $[\Theta(\hat{\Omega})]$. The second skew-spectrum is constructed by cross-correlating the squared map $[\Theta^2(\hat{\Omega})]$ against $[\nabla^2\Theta(\hat{\Omega})]$. Analogously the third skew-spectrum represents the cross-spectra that can be constructed using $[\nabla\Theta(\hat{\Omega}) \cdot \nabla\Theta(\hat{\Omega})]$ and $[\nabla^2\Theta(\hat{\Omega})]$ maps.

$$S_l^{(0)} \equiv \frac{1}{12\pi\sigma_0^4} S_l^{(\Theta^2, \Theta)} \equiv \frac{1}{12\pi\sigma_0^4} \frac{1}{2l+1} \sum_m \text{Real}([\Theta]_{lm} [\Theta^2]_{lm}^*) = \frac{1}{12\pi\sigma_0^4} \sum_{l_1 l_2} \mathcal{B}_{ll_1 l_2} J_{ll_1 l_2} W_l W_{l_1} W_{l_2} \quad (27)$$

$$\begin{aligned} S_l^{(1)} &\equiv \frac{1}{16\pi\sigma_0^2\sigma_1^2} S_l^{(\Theta^2, \nabla^2\Theta)} \equiv \frac{1}{16\pi\sigma_0^2\sigma_1^2} \frac{1}{2l+1} \sum_m \text{Real}([\nabla^2\Theta]_{lm} [\Theta^2]_{lm}^*) \\ &= \frac{1}{16\pi\sigma_0^2\sigma_1^2} \sum_{l_i} \left[l(l+1) + l_1(l_1+1) + l_2(l_2+1) \right] \mathcal{B}_{ll_1 l_2} J_{ll_1 l_2} W_l W_{l_1} W_{l_2} \end{aligned} \quad (28)$$

$$\begin{aligned} S_l^{(2)} &\equiv \frac{1}{8\pi\sigma_1^4} S_l^{(\nabla\Theta \cdot \nabla\Theta, \nabla^2\Theta)} \equiv \frac{1}{8\pi\sigma_1^4} \frac{1}{2l+1} \sum_m \text{Real}([\nabla\Theta \cdot \nabla\Theta]_{lm} [\nabla^2\Theta]_{lm}^*) \\ &= \frac{1}{8\pi\sigma_1^4} \sum_{l_i} \left[[l(l+1) + l_1(l_1+1) - l_2(l_2+1)] l_2(l_2+1) + \text{cyc.perm.} \right] \mathcal{B}_{ll_1 l_2} J_{ll_1 l_2} W_l W_{l_1} W_{l_2} \end{aligned} \quad (29)$$

$$J_{l_1 l_2 l_3} \equiv \frac{I_{l_1 l_2 l_3}}{2l_3+1} = \sqrt{\frac{(2l_1+1)(2l_2+1)}{(2l_3+1)4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}. \quad (30)$$

$$S^{(i)} = \sum_l (2l+1) S_l^{(i)} \quad (31)$$

$$\sigma_j^2 = \frac{1}{4\pi} \sum_l (2l+1) [l(l+1)]^j \mathcal{C}_l W_l^2 \quad (32)$$

This set of equations constitutes one of the main results of this paper. The matrices here denote the Wigner $3j$ symbols Edmonds (1968), W_l represents the smoothing window which can be, e.g., top-hat or Gaussian or compensated. Each of these spectra probes the same bispectrum $\mathcal{B}_{ll_1 l_2}$ but with different weights for individual triplets of modes $(l, l_1 l_2)$ specifying a triangle in the harmonic domain. The skew-spectrum sums over all possible configuration of the bispectrum keeping one of its side l fixed. For each individual choice of l we can compute the skew-spectra

$S_l^{(i)}$. The extraction of skew-spectra from data is relatively straight forward. It consists of construction of the relevant maps in real space either by algebraic or differential operation and then cross-correlating them in the multipole domain. Issues related to mask and noise will be dealt with in later sections. We will show that even in the presence of mask the computed skew spectra can be inverted to give a unbiased estimate of all-sky skew-spectra. Presence of noise will only affect the scatter. We have explicitly displayed the experimental beam b_l in all our expressions.

To derive the above expressions, we first express the spherical harmonic expansion of fields the $[\nabla^2\Theta(\hat{\Omega})]$, $[\nabla\Theta(\hat{\Omega}) \cdot \nabla\Theta(\hat{\Omega})]$ and $[\Theta^2(\hat{\Omega})]$ in terms of the harmonics of the original fields Θ_{lm} . These expressions involve the $3j$ functions as well as factors that depend on various l_i dependent weight factors.

$$\begin{aligned} [\nabla^2\Theta(\hat{\Omega})]_{lm} &= \int d\hat{\Omega} Y_{lm}^*(\hat{\Omega}) [\nabla^2\Theta(\hat{\Omega})] = -l(l+1)\Theta_{lm} \\ [\Theta^2(\hat{\Omega})]_{lm} &= \int d\hat{\Omega} Y_{lm}^*(\hat{\Omega}) [\Theta^2(\hat{\Omega})] = \sum_{l_i m_i} (-1)^m \Theta_{l_1 m_1} \Theta_{l_2 m_2} I_{l_1 l_2 l} \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & -m \end{pmatrix}. \\ [\nabla\Theta(\hat{\Omega}) \cdot \nabla\Theta(\hat{\Omega})]_{lm} &= \int d\hat{\Omega} Y_{lm}^*(\hat{\Omega}) [\nabla\Theta(\hat{\Omega}) \cdot \nabla\Theta(\hat{\Omega})] = \sum_{l_i m_i} \Theta_{l_1 m_1} \Theta_{l_2 m_2} \int d\hat{\Omega} Y_{lm}^*(\hat{\Omega}) [\nabla Y_{l_1 m_1}(\hat{\Omega}) \cdot \nabla Y_{l_2 m_2}(\hat{\Omega})] \end{aligned} \quad (33)$$

$$\begin{aligned} &= \frac{1}{3} \sum_{l_i m_i} [l_1(l_1+1) + l_2(l_2+1) - l(l+1)] \int d\hat{\Omega} Y_{lm}^*(\hat{\Omega}) Y_{l_1 m_1}(\hat{\Omega}) Y_{l_2 m_2}(\hat{\Omega}) \\ &= \frac{1}{3} \sum_{l_i m_i} (-1)^m [l_1(l_1+1) + l_2(l_2+1) - l(l+1)] \Theta_{l_1 m_1} \Theta_{l_2 m_2} I_{l_1 l_2 l} \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & -m \end{pmatrix}. \end{aligned} \quad (34)$$

We can define the power spectrum associated with the MFs through the following third order expression:

$$V_k^{(3)} = \sum_l [V_k]_l (2l+1) = \frac{1}{6} \sum_l (2l+1) \left\{ S_l^{(0)} H(\nu) + \frac{k}{3} S_l^{(1)} H(\nu) + \frac{k(k-1)}{6} S_l^{(2)} H_{k-2}(\nu) + \dots \right\}. \quad (35)$$

The ‘‘three-skewness’’ defines the triplets of Minkowski Functionals (MFs). At the level of two-point statistics, in the harmonic domain we have three power-spectra associated with Minkowski-Functional $V_k^{(3)}$ that depend on the three skew-spectra we have defined. We will show later in this paper that the fourth order correction terms also have a similar form, but with an additional monopole contribution that can be computed from the lower order one-point terms such as the three skewness defined here. The result presented here is important and implies that we can study the contributions to each of the MFs $v_k(\nu)$ as a function of harmonic mode l . This is especially significant result as various form of non-Gaussianity will have different l dependence and hence they can potentially be distinguished. The ordinary MFs add contributions from all individual l modes and hence have less power in differentiating various contributing sources of non-Gaussianity. This is one of main motivations for extending the concept of MFs (single numbers) to one-dimensional objects similar to power spectrum. It is worth pointing out that the skewness, along with the generalized skewness parameters, are typically less sensitive to the background cosmology. They are more sensitive to the underlying model of non-Gaussianity. The main dependence on cosmology typically results from the normalization coefficients such as σ_0 and σ_1 which are determined using the power spectrum of tSZ i.e. Θ .

In Figure-3 we plot the three skew spectra for a range of Gaussian beam with FWHM = $0' - 25'$. We consider an ideal all-sky experimental set up without any instrumental noise. The analytical results are computed using a halo model prediction for the bispectrum. The corresponding skew-spectra that include Planck type noise are displayed in Figure-4. As expected the effect of noise is more pronounced at smaller angular scales. In Figure-5 ideal noise free all-sky results are plotted for the diffuse component. The bispectrum in this case is constructed using the perturbative approach. The Figure-6 shows the corresponding results for a Planck type all-sky experiments. Notice that for near all-sky experiments the signal-to-noise will degrade as f_{sky} . All results shown are for $f_{sky} = 1$.

5 ESTIMATORS AND THEIR SCATTER

As we have noted before, the estimators for the skew-spectra can be most easily computed by cross-correlating maps in the harmonic domain. These maps are constructed in real space by applying various derivative operators. In the presence of mask the recovered skew-spectra will depend on the mask. The presence of mask typically introduces a mode-mode coupling. The approach we adopt here to reconstruct the unbiased power

spectra in the presence of mask exploits the so-called Pseudo- \mathcal{C}_l (Hivon et al. 2002), which involves expressing the observed power spectra \mathcal{C}_l in the presence of a mask as a linear combination of unbiased all-sky power spectra.

The three different generalized skew spectra that we have introduced here can be thought as cross-spectra of relevant fields. We denote these generic fields by A and B and will denote the generic skew-spectra as $S_l^{A,B}$. The skew-spectra that is recovered in the presence of mask will be represented as $\tilde{S}_l^{A,B}$ and the unbiased estimator will be denoted by $\hat{S}_l^{A,B}$. The skew-spectra recovered in the presence of mask $\tilde{S}_l^{A,B}$ will be biased. However to construct an unbiased estimator $\hat{S}_l^{A,B}$ for the skew-spectra the following procedure is sufficient. The derivation follows the same arguments as detailed in Munshi et al. (2009) and will not be reproduced here.

$$\tilde{S}_l^{A,B} = \frac{1}{2l+1} \sum_m \tilde{A}_{lm} \tilde{B}_{lm}^*; \quad \tilde{S}_l^{A,B} = \sum_{l'} M_{ll'} S_l^{A,B}; \quad M_{ll'} = \frac{1}{2l+1} \sum_{l''} I_{ll'l''}^2 |w_{l''}|^2; \quad \{A, B\} \in \{\Theta, \Theta^2, (\nabla\Theta \cdot \nabla\Theta), \nabla^2\Theta\}. \quad (36)$$

In this notation we can write the skewness parameters defined previously as:

$$S^{(0)} \equiv X_{(0)} S_l^{(\Theta^2, \Theta)}; \quad S^{(1)} \equiv X_{(1)} S_l^{(\Theta^2, \nabla^2\Theta)}; \quad S^{(2)} \equiv X_{(2)} S_l^{(\nabla\Theta \cdot \nabla\Theta, \nabla^2\Theta)} \quad \{X_{(0)}, X_{(1)}, X_{(2)}\} \equiv \left\{ \frac{1}{12\pi\sigma_0^4}, \frac{1}{16\pi\sigma_0^2\sigma_1^2}, \frac{1}{8\pi\sigma_1^4} \right\} \quad (37)$$

The mode-mode coupling matrix M is constructed from the power spectra of the mask $w_{l''}$ and used for estimation of unbiased skew-spectra $\hat{S}_l^{A,B}$. Typically the mask consists of bright stars and saturated spikes where no lensing measurements can be performed. The results that we present here are generic. The estimator thus constructed is an unbiased estimator. The computation of covariance of the scatter in the estimates can be computed using analytical methods, thereby avoiding the need of expensive Monte-Carlo simulations. The scatter or covariance of the unbiased estimates $\langle \delta\hat{S}_l^{A,B} \delta\hat{S}_{l'}^{A,B} \rangle$ is related to that of the direct estimates $\langle \delta\tilde{S}_l^{A,B} \delta\tilde{S}_{l'}^{A,B} \rangle$ from the masked sky by a similarity transformation. The transformation is given by the same mode coupling matrix M .

$$\hat{S}_l^{A,B} = \sum_{l'} [M^{-1}]_{ll'} \tilde{S}_{l'}^{A,B}; \quad \langle \delta\hat{S}_l^{A,B} \delta\hat{S}_{l'}^{A,B} \rangle = \sum_{LL'} M_{lL}^{-1} \langle \delta\tilde{S}_L^{A,B} \delta\tilde{S}_{L'}^{A,B} \rangle M_{L'l'}^{-1}; \quad \langle \hat{S}_l^{A,B} \rangle = S_l^{A,B}; \quad \{A, B\} \in \{\Theta, \Theta^2, (\nabla\Theta \cdot \nabla\Theta), \nabla^2\Theta\}. \quad (38)$$

The power spectra associated with the MFs are linear combinations of the skew-spectra (see Eq.(26)). In our approach the power spectra associated with the MFs are secondary and can be constructed using the skew-spectra that are estimated directly from the data.

The construction of an estimator is incomplete without an estimate of its variance. The variance or the scatter in certain situation is computed using Monte-Carlo (MC) simulations which are computationally expensive. In our approach, it is possible to compute the covariance of our estimates of various S_{ls} , i.e. $\langle \delta S_l \delta S_{l'} \rangle$ under the same simplifying assumptions that higher order correlation functions can be approximated as Gaussian. This allows us to express the error covariance in terms of the relevant power spectra. The generic expression can be written as:

$$[\hat{V}_k^{(2)}]_l = \sum_{l'} [M^{-1}]_{ll'} [\tilde{V}_k^{(2)}]_{l'}; \quad \langle \delta\hat{V}_k^{(2)} \delta\hat{V}_{k'}^{(2)} \rangle = \sum_{LL'} M_{lL}^{-1} \langle \delta[\tilde{V}_k^{(2)}]_l \delta[\tilde{V}_{k'}^{(2)}]_{l'} \rangle M_{L'l'}^{-1} \quad (39)$$

We would like to point out here that, in case of limited sky coverage, it may not be possible to estimate the skew-spectra mode by mode as the mode coupling matrix may become singular and a broad binning of the spectra may be required.

$$\langle [\delta S_l^{X,Y}] \delta S_{l'}^{X,Y} \rangle = f_{sky}^{-1} \frac{1}{2l+1} \left[C_l^{X,X} C_{l'}^{Y,Y} + [S_l^{X,Y}]^2 \right] \delta_{ll'}; \quad \{X, Y\} \in \{\Theta, \Theta^2, \nabla\Theta(\hat{\Omega}) \cdot \nabla\Theta(\hat{\Omega}), \nabla^2\Theta(\hat{\Omega})\}. \quad (40)$$

Here the fraction of sky covered by the survey is denoted by f_{sky} . The expressions for the skew-spectra are quoted in $S_l^{(\Theta^2, \Theta)}$, $S_l^{(\kappa^2, \nabla^2\kappa)}$ and $S_l^{(\nabla\Theta \cdot \nabla\Theta, \nabla^2\Theta)}$ are given in Eq.(32). The expressions for covariance also depend on a set of power spectra i.e. $S_l^{(\Theta^2, \Theta^2)}$, $S_l^{(\nabla^2\Theta, \nabla^2\Theta)}$, $S_l^{(\nabla\Theta \cdot \nabla\Theta, \nabla^2\Theta)}$ and $S_l^{\Theta, \Theta}$. These are given by the following expression:

$$C_l^{\nabla \cdot \nabla, \nabla \cdot \nabla} = \sum_{l_1 l_2} C_{l_1}^t C_{l_2}^t [l_1(l_1+1) + l_2(l_2+1) - l(l+1)]^2 I_{ll_1 l_2}^2; \quad C_l^{[\Theta^2, \Theta^2]} = \sum_{l_1 l_2} C_{l_1}^t C_{l_2}^t I_{ll_1 l_2}^2; \quad C_l^{[\nabla^2\Theta, \nabla^2\Theta]} = l^2(l+1)^2 C_l^t \quad (41)$$

Here C_l^t includes signal and noise power spectra $C_l^t = C_l^S b_l^2 + C_l^N$. The experimental beam is denoted by b_l . Using these equations it is possible to compute the scatter in various skew-spectra. The scatter will depend on the skew-spectra as well as various cross-spectra of various combination

Signal to Noise (S/N): Halo Model Results

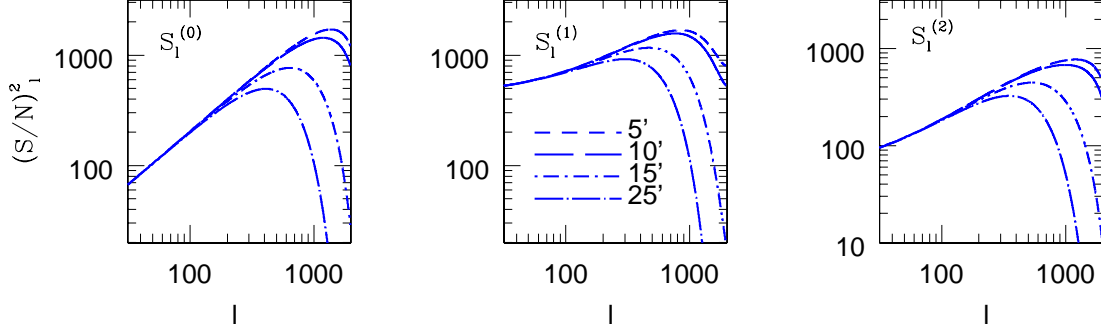


Figure 7. The signal-to-noise (S/N) squared for various skew-spectra $S_l^{(0)}$ (left panel), $S_l^{(1)}$ (middle panel) and $S_l^{(2)}$ (right panel) are plotted as a function of angular harmonics l . We choose four different Gaussian beams with FWHM $\theta_s = 0', 5', 15', 25'$. The higher resolution (smaller FWHM) reaches higher signal-to-noise. The plots correspond to an ideal experiment with all-sky coverage and no instrumental noise. The expressions used for computation of the signal-to-noise are given in Eq.(45)-Eq.(47). The results correspond to an ideal all-sky $f_{sky} = 1$ no-noise setup. For near all-sky experiments the signal to noise will scale directly with f_{sky} .

of product fields constructed from the original tSZ maps.

$$\langle \delta S_l^{[\Theta^2, \Theta]} \delta S_l^{[\Theta^2, \Theta]} \rangle = f_{sky}^{-1} \frac{1}{2l+1} \left[C_l^{[\Theta^2, \Theta^2]} C_l^{[\Theta, \Theta]} + [S_l^{[\Theta^2, \Theta]}]^2 \right] \quad (42)$$

$$\langle \delta S_l^{[\Theta^2, \nabla^2 \Theta]} \delta S_l^{[\Theta^2, \nabla^2 \Theta]} \rangle = f_{sky}^{-1} \frac{1}{2l+1} \left[C_l^{[\Theta^2, \Theta^2]} C_l^{[\nabla \cdot \nabla, \nabla \cdot \nabla]} + [S_l^{[\Theta^2, \nabla^2 \Theta]}]^2 \right] \quad (43)$$

$$\langle \delta S_l^{[\nabla \Theta \cdot \nabla \Theta, \nabla^2 \Theta]} \delta S_l^{[\nabla \Theta \cdot \nabla \Theta, \nabla^2 \Theta]} \rangle = f_{sky}^{-1} \frac{1}{2l+1} \left[C_l^{[\nabla^2 \Theta, \nabla^2 \Theta]} C_l^{[\nabla \cdot \nabla, \nabla \cdot \nabla]} + [S_l^{[\nabla \Theta \cdot \nabla \Theta, \nabla^2 \Theta]}]^2 \right] \quad (44)$$

The cumulative signal to noise up to a given l_{max} using these expression for estimators $S^{(0)}$ can now be expressed as:

$$\left[(S/N)_{l_{max}}^{(0)} \right]^2 = f_{sky} \sum_{l=2}^{l_{max}} (2l+1) (S_l^{[\Theta^2, \Theta]})^2 / \left[X_{(0)}^2 C_l^{[\Theta^2, \Theta^2]} C_l^{[\Theta, \Theta]} + (S_l^{[\Theta^2, \Theta]})^2 \right] \quad (45)$$

$$\left[(S/N)_{l_{max}}^{(1)} \right]^2 = f_{sky} \sum_{l=2}^{l_{max}} (2l+1) (S_l^{[\Theta^2, \nabla^2 \Theta]})^2 / \left[X_{(1)}^2 C_l^{[\Theta^2, \Theta^2]} C_l^{[\nabla^2, \nabla^2]} + [S_l^{[\Theta^2, \nabla^2 \Theta]}]^2 \right] \quad (46)$$

$$\left[(S/N)_{l_{max}}^{(2)} \right]^2 = f_{sky} \sum_{l=2}^{l_{max}} (2l+1) (S_l^{[\nabla \Theta \cdot \nabla \Theta, \nabla^2 \Theta]})^2 / \left[X_{(2)}^2 C_l^{[\nabla^2 \Theta, \nabla^2 \Theta]} C_l^{[\nabla \cdot \nabla, \nabla \cdot \nabla]} + [S_l^{[\nabla \Theta \cdot \nabla \Theta, \nabla^2 \Theta]}]^2 \right] \quad (47)$$

For the study of the tSZ effect we find that a robust determination of the generalised skew-spectra is possible for individual modes for almost the entire range of l values we have probed. This is important in differentiating them from various other sources of non-Gaussianity. The skew-spectra are integrated statistics, meaning that their value at a given l depends on the entire range of l values probed. These results can also be extended to take into account the cross-correlation among various skew-spectra extracted from the same data.

$$\langle \delta S_l^{[\Theta^2, \Theta]} \delta S_l^{[\Theta^2, \nabla^2 \Theta]} \rangle = f_{sky}^{-1} \frac{1}{2l+1} \left[C_l^{[\Theta^2, \Theta^2]} C_l^{[\Theta, \nabla^2 \Theta]} + S_l^{[\Theta^2, \nabla^2 \Theta]} S_l^{[\Theta, \Theta^2]} \right] \quad (48)$$

$$\langle \delta S_l^{[\Theta^2, \Theta]} \delta S_l^{[\nabla \Theta \cdot \nabla \Theta, \nabla^2 \Theta]} \rangle = f_{sky}^{-1} \frac{1}{2l+1} \left[S_l^{[\Theta^2, \nabla^2 \Theta]} C_l^{[\Theta, \nabla \Theta \cdot \nabla \Theta]} + C_l^{[\Theta^2, \nabla \Theta \cdot \nabla \Theta]} C_l^{[\Theta, \nabla^2 \Theta]} \right] \quad (49)$$

$$\langle \delta S_l^{[\Theta^2, \nabla^2 \Theta]} \delta S_l^{[\nabla \Theta \cdot \nabla \Theta, \nabla^2 \Theta]} \rangle = f_{sky}^{-1} \frac{1}{2l+1} \left[C_l^{[\Theta^2, \nabla \Theta \cdot \nabla \Theta]} C_l^{[\nabla^2 \Theta, \nabla^2 \Theta]} + S_l^{[\Theta^2, \nabla^2 \Theta]} S_l^{[\nabla^2 \Theta, \nabla \Theta \cdot \nabla \Theta]} \right] \quad (50)$$

This discussion involves the lowest order departure from Gaussianity in MFs using a third order statistic, namely the bispectrum. The next to

Signal to Noise (S/N): Halo Model Results

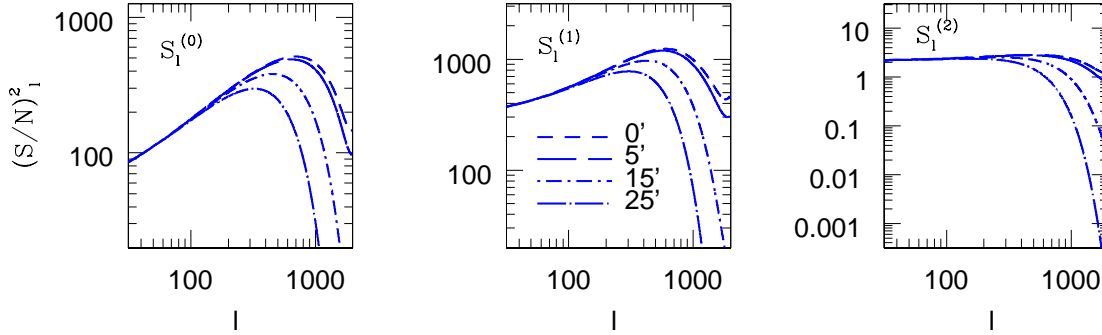


Figure 8. Same as previous figure but with Planck noise added. Inclusion of noise degrades the signal-to-noise. The effect is most prominent for $S_l^{(2)}$ which is related to the fact that $S_l^{(2)}$ gives more weights to smaller scales than other estimates and hence more affected by the noise.

Signal to Noise (S/N): Perturbative Results

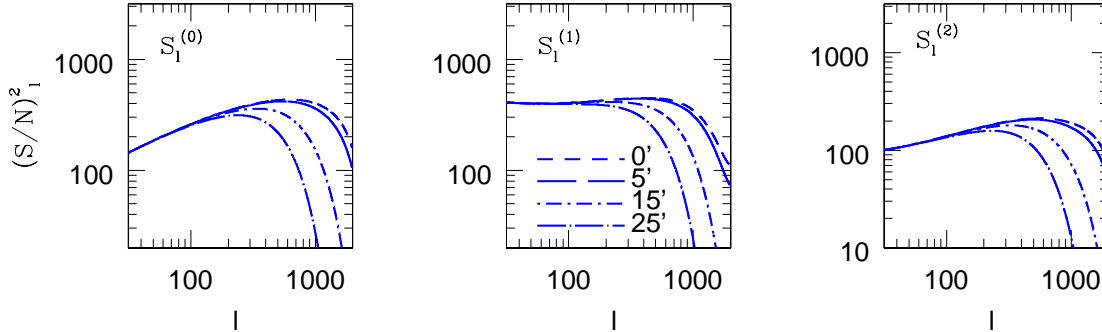


Figure 9. Same as Figure-8 but for perturbative calculations. The perturbative bispectrum has less power at higher l compared to halo model calculations.

leading descriptions are characterized by the trispectrum which is a fourth order statistics. It is possible to extend the definition of skew-spectra to the case of kurt-spectra or the power spectrum associated with tri-spectra. The power spectra associated with the Minkowski Functionals can be defined completely up to fourth order using the skew- and the kurt-spectra. However the corrections to leading order statistics from kurt-spectra are sub-dominant and leading order terms are enough to study the departure from Gaussianity. In any case it is nevertheless straight forward to implement an estimator which will estimate the power-spectrum associated with the MFs from a noisy data by including both third order and fourth order statistics is relatively straight forward. The issues has been dealt with in detail in (Munshi, Smidt & Cooray 2010) in the context of CMB sky.

In addition to the three generalized skew-spectra that define the MFs at lowest order in non-Gaussianity, it is possible also to construct additional skew-spectra that work with different set of weights. In principle an arbitrary number of such skew-spectra can be constructed though they will not have direct links with the morphological properties that we have focused on, in this paper they can still be used as a source of independent information on the bispectrum and can be used in principle to separate sources of non-Gaussianity, whether it be primordial or induced by late-time gravitational effects. In Figure-7 we have plotted the signal to noise for the three skew-spectra for different choice of the FWHM as indicated for an ideal experiment. The analytical results for the underlying bispectrum is obtained using halo model prescription. The left, middle and right panel correspond to $S_l^{(0)}$, $S_l^{(1)}$ and $S_l^{(2)}$ respectively. In Figure-8 the signal-to-noise for an all-sky experiment with Planck

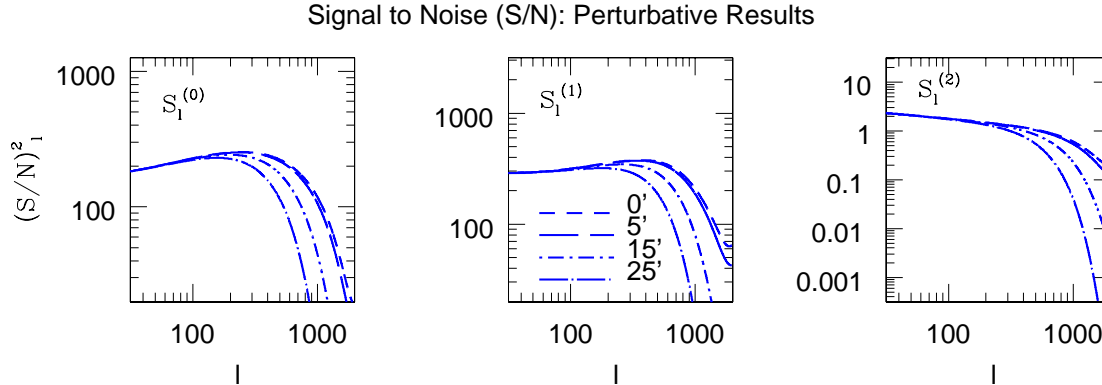


Figure 10. These results correspond to perturbative analysis but with instrumental noise for a Planck-type experiment included.

type experiment is shown. As expected the with smaller beam size will have higher signal-to-noise. In Figure-9 results for Planck type experiment is shown but using perturbative calculations and finally we show perturbative results that include noise in Figure-9. The skew-spectra $S_l^{(2)}$ has higher signal-to-noise among the three skew-spectra. The very high signal to noise will allow an accurate determination of all the three spectra especially $S_l^{(0)}$ and $S_l^{(1)}$ for the entire range of l values probed.

6 CONCLUSION

In this study we have examined the prospects for extracting non-Gaussianity statistics from CMB surveys using MFs to probe the non-Gaussianity that originates from the tSZ effect. The tSZ effect is associated with the hot gas in large-scale structure that is detectable by multi-frequency experiments, in particular Planck.. When compared with the CMB temperature anisotropies the tSZ effect has a distinct spectral signature which means it can be effectively separated from the primary CMB contributions. This will provide an unique opportunity to probe tSZ effect using frequency-cleaned data in the very near future. In the past, the statistical analysis of the frequency cleaned tSZ maps has so far been mainly focused on lower order statistics. Morphological studies involving the MFs are complementary to the lower order statistics and carry independent information. In this paper we have generalized the study of MFs that are typically used in various other context in cosmology to the case of tSZ maps.

As previously mentioned, the tSZ effect traces the fluctuations associated with the large scale distribution of the baryonic gas. The generation of pressure fluctuation in the virialized dark matter halos can be modeled by assuming them to be in hydrostatic equilibrium with the dark matter distribution within the halo. The shock heated gas typically correspond to over-densities in excess of $\delta \geq 200$. The unshocked photoionized baryonic gas typically traces the large scale distribution of the dark matter distribution. The typical over-density in unshocked photoionised baryons is typically $\delta \leq 10$.

We have modelled the tSZ effect using two independent techniques. For smaller overdensities we employ a simplistic biasing model that rely on a perturbative description of dark matter clustering. The statistics of the hot gas is than linked to that of the dark matter using a linear (redshift dependent) biasing scheme. While this type of modelling is adequate for small overdensities a more elaborate analytical scheme is required for a detailed description of baryonic clustering at smaller scales. We consider a halo model based approach for the gas in collapsed virialised halos. The specific number density and radial profile of these halos are modeled using Press-Sechter formalism or its variants. These two pictures of baryonic clustering is complementary to each other. The tSZ angular power spectrum corresponds to the projected power spectrum of the baryonic pressure fluctuation and the tSZ angular bispectrum correspond to the bispectrum of pressure fluctuations projected onto the surface of the sky.

Few comments on the validity of the perturbative results are in order. The tSZ power spectrum by and large depends on the one halo term in the halo modelling. However in the perturbative regime we are mostly probing the Jean-Scale smoothed gas that is not in collapsed objects and traces the smoothed large scale dark matter distribution. To probe the large scale SZ effect removing X-ray bright clusters can reduce contributions from collapsed halos. Thus the effect captured by the linear biasing scheme should be understood as a signal in blank fields where such clusters are

absent. It provides the lower limit of SZ effect from large scale structure distribution. Though the diffuse component of the tSZ effect is beyond WMAP detection threshold the situation may improve with future data sets such as Planck (Hansen et al. 2005).

The tSZ effect is intrinsically non-Gaussian. While the tSZ power spectrum is sensitive to the amplitude of the density fluctuations the higher order statistics such as skewness of tSZ effect can be used to separate the pressure bias from the amplitude of the density fluctuations. The skewness is related to the bispectrum of the tSZ effect. The individual modes of a specific bispectrum is defined by the triplets of the harmonic numbers (l_1, l_2, l_3) . However individual modes of the bispectrum has low signal to noise and may not be easy to estimate from a noisy data. In a recent study Munshi & Heavens (2010) advocated using a spectra correspond to the higher order spectra in general and bispectrum in particular. The skew spectrum is the lowest such spectrum that is constructed from bispectrum. The skew-spectra can retain some of the shape dependence in bispectrum without compressing it to a single number. The three different skew-spectra that we introduced in this paper are generalizations of the ordinary skew-spectrum that was originally introduced in Munshi & Heavens (2010) and later used in the context of weak lensing and CMB for morphological analysis. These three different skew-spectra can also be used to study the morphology of the tSZ maps as described by the MFs. They associate varying weights to individual bispectrum modes when constructing the skew-spectra and carry independent information.

Using near all-sky setup and noise that reflects ongoing CMB observations such as Planck we study how well the three different skew-spectra can be estimated from the data. We find that the data will allow a *robust* determination of two of the three skew-spectra that we have considered with a very high level of signal to noise. This is true for both perturbative results and the results that are based on halo model for the entire range of smoothing angular scales that we have studied. We also find the estimation of $S^{(2)}$ will be dominated by noise. The high signal-to-noise for the other two power spectra will allow mode by mode estimation of each skew-spectra. This can help to differentiate them from other sources of non-Gaussianity.

The method that we pursue here depends on frequency cleaned tSZ maps. The tSZ effect can also be studied using cross-correlation techniques that involve external tracers. Such methods typically employ mixed bispectra. The results however lack the frequency information and typically confusion noise dominated. The study of tSZ using bispectrum from frequency cleaned maps typically provide additional signal-to-noise due to the frequency information. In the absence of frequency information the background CMB plays the role of intrinsic noise that degrades the signal-to-noise ratio. It is also interesting to note that removal of tSZ from the CMB maps may actually help in detection of other sub-dominant effects that we have not studied here, such as the kinetic Sunyaev Zeldovich effect (kSZ).

Some of these techniques described here will have wider applicability. The idea of generalized skew-spectra was shown to be useful in the context of weak lensing surveys (Munshi et al. 2011) and probing primordial non-Gaussianity from CMB maps (Munshi, Smidt & Cooray 2010). In this study we have concentrated on only the leading order terms in the construction of the power spectra associated with the MFs. However the next to leading order terms can also be taken into account using the same formalism. The next to leading order terms will involve kurt-spectra that generalizes the concept of skew-spectra at fourth order. The four quadruplet of generalized kurt-spectra are related to the trispectra in a way that is similar to the relationship of skew-spectra and the bispectrum we have considered in this paper. The generalized kurt-spectra and the related generalized kurtosis can also be extracted from the data using the same PCL approach which we have discussed here. However the correction to generalised skew-spectra associated with MFs resulting from the trispectrum are expected to be negligible compared to the leading contribution from the generalized skew-spectra.

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